BMO 2008–2009 Round 1 Problem 1—Generalisation

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December 2008

1 Introduction

Problem 1 is:

> 1. Consider a standard 8×8 chessboard consisting of 64 small squares coloured in the usual pattern, so 32 are black and 32 are white. A *zig-zag* path across the board is a collection of eight white squares, one in each row, which meet at their corners. How many zig-zag paths are there?

This problem generalises to any rectangular board. If both the number of rows and the number of columns are odd, it matters whether the colour specified is the colour of all the corner squares or none of them; otherwise, symmetry considerations show that the numbers for each colour are equal. Thus we consider the problem:

> Let n and r be positive integers, and consider a rectangular chessboard with r rows of n squares each. A *zig-zag* path across the board is a collection of r squares, one in each row, which meet at their corners. Let $m_{n,r}$ be the number of zig-zag paths, $c_{n,r}$ be the number of zig-zag paths whose colour is that of the square in the top right corner of the board, and $n_{n,r}$ be the number of zig-zag paths whose colour is the opposite of that of the square in the top right corner of the board. Determine $m_{n,r}$, $c_{n,r}$ and $n_{n,r}$.

Clearly $m_{n,r} = c_{n,r} + n_{n,r}$, and as noted above $c_{n,r} = n_{n,r}$ if n or r is even.

The "standard" method of solving the original problem is simply to count the number of zig-zag paths from the top row to each square in each row in turn, writing down 1s in the top row and then making each value in each subsequent row the sum of values above it on either side, leading to the following diagram; this also readily allows computing values for other rectangular boards.

1		1		1		1	
	2		2		2		1
2		4		4		3	
	6		8		7		3
6		14		15		10	
	20		29		25		10
20		49		54		35	
	69		103		89		35

2 Eigenvalue methods

The map above from the numbers in one row to the numbers in the next is a linear transformation given by a matrix $M = (M_{ij})$, where $M_{ij} = 1$ if $|i-j| = 1$ and $M_{ij} = 0$ otherwise.

As a real symmetric matrix, this has real eigenvalues λ_k and an orthonormal basis of eigenvectors $\mathbf{v}_k$. The sum of the elements in the final row will be

$$\mathbf{u}^T M^{r-1}\mathbf{v} = \sum_{k=1}^{n} (\mathbf{u}\cdot\mathbf{v}_k)\lambda_k^{r-1}(\mathbf{v}\cdot\mathbf{v}_k),$$

where $\mathbf{u}$ is the all-1s vector and $\mathbf{v}$ is the vector of values in the first row.

We claim that $\lambda_k = 2\cos\frac{\pi k}{n+1}$ and that $\mathbf{v}_k$ is the vector with s^{th} coordinate $v_{k,s} = \sqrt{2/(n+1)}\sin\frac{\pi k s}{n+1}$. Certainly this is an eigenvector with the given eigenvalue, since

$$\sin\left(\tfrac{\pi k s}{n+1} - \tfrac{\pi k}{n+1}\right) + \sin\left(\tfrac{\pi k s}{n+1} + \tfrac{\pi k}{n+1}\right) = 2\sin\tfrac{\pi k s}{n+1}\cos\tfrac{\pi k}{n+1},$$

so it remains to check the normalisation factor; that is, to show that

$$\sum_{s=1}^{n} \sin^2 \tfrac{\pi ks}{n+1} = \frac{n+1}{2}.$$

Now we have

$$\begin{aligned}
\sum_{s=1}^{n} \sin^2 \tfrac{\pi ks}{n+1} &= \frac{1}{2}\sum_{s=1}^{n}\left(1 - \cos\tfrac{2\pi ks}{n+1}\right) \\
&= \frac{n+1}{2} - \frac{1}{2}\sum_{s=0}^{n}\cos\tfrac{2\pi ks}{n+1}
\end{aligned}$$

and since

$$2\cos\tfrac{2\pi ks}{n+1}\cos\tfrac{2\pi k}{n+1} = \cos\left(\tfrac{2\pi ks}{n+1} - \tfrac{2\pi k}{n+1}\right) + \cos\left(\tfrac{2\pi ks}{n+1} + \tfrac{2\pi k}{n+1}\right)$$

we have

$$\sum_{s=0}^{n}\cos\tfrac{2\pi ks}{n+1} = \cos\tfrac{2\pi k}{n+1}\sum_{s=0}^{n}\cos\tfrac{2\pi ks}{n+1}$$

from which the result follows as $\cos\tfrac{2\pi k}{n+1} \neq 1$. (The original sum may also be evaluated directly by expressing it in terms of geometric series in $\exp\left(\tfrac{\pi ik}{n+1}\right)$.)

It remains to evaluate the dot products above for particular $\mathbf{u}$ and $\mathbf{v}$. (Again, this could be done directly with geometric series, though here we demonstrate the results without this.) For even k, we have $\mathbf{u}\cdot\mathbf{v}_k = 0$ as positive and negative terms cancel. So suppose k is odd. First, we claim that

$$\mathbf{v}\cdot\mathbf{v}_k = \sqrt{2/(n+1)}\frac{\sin\frac{\pi k}{n+1}}{1-\cos\frac{\pi k}{n+1}}.$$

For, put

$$S = \sum_{s=1}^{n}\sin\frac{\pi ks}{n+1}.$$

We then have

$$\begin{aligned}
(1-\cos\tfrac{\pi k}{n+1})S &= \sum_{s=1}^{n}\left(\sin\tfrac{\pi ks}{n+1} - \cos\tfrac{\pi k}{n+1}\sin\tfrac{\pi ks}{n+1}\right) \\
&= \sum_{s=1}^{n}\left(\sin\tfrac{\pi ks}{n+1} - \tfrac{1}{2}\sin\tfrac{\pi k(s+1)}{n+1} - \tfrac{1}{2}\sin\tfrac{\pi k(s-1)}{n+1}\right) \\
&= \tfrac{1}{2}\left(\sin\tfrac{\pi k}{n+1} - \sin 0 + \sin\tfrac{\pi kn}{n+1} - \sin\pi k\right) \\
&= \sin\tfrac{\pi k}{n+1}.
\end{aligned}$$

Now

$$
\begin{aligned}
(\mathbf{v}\cdot\mathbf{v}_k)^2 &= \frac{2}{n+1}\frac{\sin^2\frac{\pi k}{n+1}}{(1-\cos\frac{\pi k}{n+1})^2} \\
&= \frac{2}{n+1}\frac{1-\cos^2\frac{\pi k}{n+1}}{(1-\cos\frac{\pi k}{n+1})^2} \\
&= \frac{2}{n+1}\frac{1+\cos\frac{\pi k}{n+1}}{1-\cos\frac{\pi k}{n+1}}
\end{aligned}
$$

so we conclude that

$$
m_{n,r} = \frac{2}{n+1}\sum_{\substack{k=1\\ k\ \text{odd}}}^{n} \left(2\cos\tfrac{\pi k}{n+1}\right)^{r-1}\left(\frac{1+\cos\frac{\pi k}{n+1}}{1-\cos\frac{\pi k}{n+1}}\right)
$$

with

$$
c_{n,r} = n_{n,r} = \frac{1}{n+1}\sum_{\substack{k=1\\ k\ \text{odd}}}^{n} \left(2\cos\tfrac{\pi k}{n+1}\right)^{r-1}\left(\frac{1+\cos\frac{\pi k}{n+1}}{1-\cos\frac{\pi k}{n+1}}\right)
$$

if either n or r is even.

Similarly, if we consider the vectors $\mathbf{v}$ arising from restricting to a particular colour of squares when both n and r are odd, we find

$$
c_{n,r} = \frac{2}{n+1}\sum_{\substack{k=1\\ k\ \text{odd}}}^{n} \left(2\cos\tfrac{\pi k}{n+1}\right)^{r-1}\left(\frac{1}{1-\cos\frac{\pi k}{n+1}}\right)
$$

and

$$
n_{n,r} = \frac{2}{n+1}\sum_{\substack{k=1\\ k\ \text{odd}}}^{n} \left(2\cos\tfrac{\pi k}{n+1}\right)^{r-1}\left(\frac{\cos\frac{\pi k}{n+1}}{1-\cos\frac{\pi k}{n+1}}\right)
$$

in this case.

3 Combinatorial methods for the square case

The above results by eigenvalue methods give the asymptotics of the functions $m_{n,r}$, $c_{n,r}$ and $n_{n,r}$ as $r\to\infty$ for fixed n. The original problem, however, was for a square board, $r=n$. In this case we may reason purely combinatorially.

Consider zig-zag paths on an infinitely wide extension of the given board, that start within the columns of the original board but may go outside it. Then there is a one-to-one correspondence between:

- zig-zag paths that start and end within the columns of the original board but go outside it in intermediate rows; and

- zig-zag paths that start within the columns of the original board and end outside it but not in the columns immediately adjacent to it on either side.

This correspondence is given simply by taking the first square at which the path (going from top to bottom, say) goes outside the original board, and reflecting the rest of the path in a vertical axis going through that square. (This does not work for rectangular boards taller than they are wide because then the reflections can take a square of the path outside the board on one side to a square outside the board on the other. The cylindrical model of the next section shows how this method generalises to other boards, but the resulting binomial coefficient sums do not appear particularly useful in general.)

Now, the number of paths starting and ending in given squares but possibly going outside the board is simply given by a binomial coefficient; considering how many times each such coefficient is counted positively or negatively for different starting squares, then pairing $\binom{r-1}{k}$ and $\binom{r-1}{r-1-k}$, we have for n even

$$\begin{aligned}
c_{n,n} &= n_{n,n} \\
&= \sum_{k=0}^{(n/2)-1} (4k-n+3)\binom{n-1}{k} \\
&= ((3-n)/2)2^{n-1} + 4\sum_{k=0}^{(n/2)-1} k\binom{n-1}{k} \\
&= (3-n)2^{n-2} + 4\sum_{k=1}^{(n/2)-1} (n-1)\binom{n-2}{k-1} \\
&= (3-n)2^{n-2} + 4(n-1)\tfrac{1}{2}\left(2^{n-2} - \binom{n-2}{(n-2)/2}\right) \\
&= (n+1)2^{n-2} - 2(n-1)\binom{n-2}{(n-2)/2}
\end{aligned}$$

and so

$$m_{n,n} = (n+1)2^{n-1} - 4(n-1)\binom{n-2}{(n-2)/2}.$$

Similarly, for odd n we have

$$\begin{aligned} c_{n,n} &= \left(\sum_{k=0}^{(n-3)/2} (4k-n+3)\binom{n-1}{k}\right) + \tfrac{n+1}{2}\binom{n-1}{(n-1)/2} \\ &= (n+1)2^{n-2} - (n-1)\binom{n-1}{(n-1)/2} \end{aligned}$$

and

$$\begin{aligned} n_{n,n} &= \left(\sum_{k=0}^{(n-3)/2} (4k-n+3)\binom{n-1}{k}\right) + \tfrac{n-1}{2}\binom{n-1}{(n-1)/2} \\ &= (n+1)2^{n-2} - n\binom{n-1}{(n-1)/2} \end{aligned}$$

so

$$m_{n,n} = (n+1)2^{n-1} - (2n-1)\binom{n-1}{(n-1)/2}.$$

4 The cylindrical model

Some of the complexity in understanding the linear map from one row to another appears to arise from the edge effects, where some values in the next row are copies of values from the row above and others are sums of two values.

Suppose the row a_1, a_2, …, a_n of length n is replaced by a cylindrical row of period $2n+2$ and values a_1, a_2, …, a_n, 0, $-a_n$, …, $-a_2$, $-a_1$, 0, …. Then each cylindrical row is mapped to the next with *every* value in the next row being the sum of the values above it on either side.

This generalises the argument in the previous section, by showing that the desired number is given by the number of paths that end up in the right columns mod $2n+2$, minus those that end up in columns that are wrong by more than 1 mod $2n+2$. It also explains the eigenvalues that arise. For, the map from one row to the next is the sum of two maps, one which shifts to the right and one which shifts to the left. Each of these maps clearly has as eigenvalues the $(2n+2)^{\text{th}}$ roots of unity, and an eigenvector with eigenvalue ζ for one map has eigenvalue ζ^{-1} for the other map. Each pair of roots $\zeta \neq \zeta^{-1}$ gives rise to a two-dimensional eigenspace for the sum of the two maps, whose eigenvalue $\zeta + \zeta^{-1}$ is one of the cosines previously found, and which has a one-dimensional subspace of the form required to have arisen from a row in the original problem by the transformation given above. The roots 1 and -1

each have a one-dimensional eigenspace, which does not include any nonzero vectors arising from the original problem.

5 Integer sequences

These problems give rise to numerous integer sequences, many of which are to be found in the On-Line Encyclopedia of Integer Sequences.

First there are at least six natural sequences for the square case: $m_{n,n}$, $c_{n,n}$, $n_{n,n}$, $c_{2n,2n}$, $c_{2n-1,2n-1}$, $n_{2n-1,2n-1}$. These appear in OEIS, as A102699(n), A153334(n), A153335(n), A153336(n), A153337(n), A153338(n) respectively.

For each fixed n, there are again at least six natural sequences if n is odd (and r varies), and at least two if n is even. Some sequences appear many times in OEIS with different offsets and initial terms; some may not appear with the given initial terms at all, with the OEIS entries listed below being for versions that are the same as the sequences described here from some point onwards but with no terms or different terms before that point.

First there are some sequences given by simple formulae. The OEIS entries listed are those that appear more or less canonical for the formulae, with no attempt in most cases to list versions with different initial terms.

- $m_{1,r} = c_{1,r} = c_{1,2r-1} = 0^{r-1} =$ A000007$(r-1)$
- $n_{1,r} = c_{1,2r} = n_{1,2r-1} = 0 =$ A000004(r)
- $m_{2,r} = 2 =$ A007395(r)
- $c_{2,r} = 1 =$ A000012(r)
- $m_{3,r} = (2\frac{1}{2}-\frac{1}{2}(-1)^r)2^{\lfloor r/2 \rfloor} =$ A029744$(r{+}2)$ (see also A063759, A090989, A145751)
- $c_{3,r-1} = n_{3,r} = 2^{\lfloor r/2 \rfloor} =$ A016116(r) (see also A060546, A131572)
- $c_{3,2r} = c_{3,2r-1} = n_{3,2r+1} = 2^r =$ A000079(r)
- $m_{4,r} = 2F_{r+2} =$ A118658$(r+3)$ (see also A006355, A047992, A054886, A055389, A068922, A078642, A090991, A128588)
- $c_{4,r} = F_{r+2} =$ A000045$(r+2)$

As n increases, fewer familiar sequences appear, although there are still some variants with different initial terms.

- $m_{5,r} = \mathrm{A090993}(r-1)$
- $c_{5,r} = \mathrm{A153339}(r)$
- $n_{5,r} = \mathrm{A068911}(r)$
- $c_{5,2r} = 4 \cdot 3^{r-1} = \mathrm{A003946}(r)$ (see also A025579, A027327, A052156)
- $c_{5,2r-1} = \mathrm{A080923}(r)$ (see also A005051, A026097, A083583, A118264)
- $n_{5,2r-1} = 2 \cdot 3^{r-1} = \mathrm{A025192}(r)$ (see also A008776, A027334, A099856, A110593)
- $m_{6,r} = \mathrm{A090995}(r-1)$
- $c_{6,r} = \mathrm{A090990}(r-1)$
- $m_{7,r} = \mathrm{A129639}(r+10)$
- $c_{7,r} = \mathrm{A030435}(r+1)$
- $n_{7,r} = \mathrm{A030436}(r+2)$
- $c_{7,2r} = \mathrm{A006012}(r+1)$
- $c_{7,2r-1} = \mathrm{A056236}(r)$
- $n_{7,2r-1} = \mathrm{A007052}(r)$ (see also A048580)
- $m_{8,r} = \mathrm{A153340}(r)$
- $c_{8,r} = \mathrm{A090992}(r-1)$
- $m_{9,r} = \mathrm{A153362}(r)$
- $c_{9,r} = \mathrm{A153363}(r)$
- $n_{9,r} = \mathrm{A153364}(r)$
- $c_{9,2r} = \mathrm{A153365}(r)$
- $c_{9,2r-1} = \mathrm{A153366}(r)$
- $n_{9,2r-1} = \mathrm{A153367}(r)$
- $m_{10,r} = \mathrm{A153360}(r)$
- $c_{10,r} = \mathrm{A090994}(r-1)$

- $m_{11,r} = \text{A153368}(r)$
- $c_{11,r} = \text{A153369}(r)$
- $n_{11,r} = \text{A153370}(r)$
- $c_{11,2r} = \text{A153371}(r)$
- $c_{11,2r-1} = \text{A153372}(r)$
- $n_{11,2r-1} = \text{A153373}(r)$
- $m_{12,r} = \text{A153361}(r)$
- $c_{12,r} = \text{A129638}(r+9)$
- The sequences for $13 \leq n \leq 16$ (and probably larger n, not checked) do not seem to appear in OEIS.

Of particular note appear to be the relations to sequences concerning the numbers of differential operations on $\mathbb{R}^n$, as enumerated in [1], [2] and [3]. It appears that $m_{n,r}$ counts differential operations on $\mathbb{R}^{2n-2}$ and $c_{2n,r}$ counts differential operations on $\mathbb{R}^{2n-1}$, although I do not have a proof or a direct combinatorial correspondence between the two problems, and the offsets listed in OEIS for the differention operation sequences do not seem consistent across all such sequences. (Any particular individual case may readily be verified using the recurrences given for the sequences for differential operations.)

Supposing those relations, A116183 gives a table corresponding to a complicated arrangement of the present sequences, while A127935 gives the sequence $c_{4,2}$, $m_{3,3}$, $c_{6,4}$, $m_{4,5}$, $c_{8,6}$, $m_{5,7}$, $c_{10,8}$, $m_{6,9}$, $c_{12,10}$, $m_{7,11}$, $\ldots$.

References

[1] Branko J. Malešević, *Some Combinatorial Aspects of Differential Operation Composition on the Space* $\mathbb{R}^n$, Univ. Beograd. Publ. Elektrotehn. Fak., Ser. Mat. **9** (1998), 29–33.

[2] Branko J. Malešević, *Some Combinatorial Aspects of Composition of a Set of Functions*, Novi Sad J. Math. **36** (2006), no. 1, 3–9.

[3] Branko J. Malešević and Ivana V. Jovović, *The Compositions of Differential Operations and the Gateaux Directional Derivative*, Journal of Integer Sequences **10** (2007), Article 07.8.2.